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BERNOULLI FACTORS THAT SPAN A TRANSFORMATION

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ABSTRACT

It is shown that every invertible ergodic transformation of positive entropy is spanned by three Bernoulli factors.

Let T be an invertible ergodic transformation of positive entropy acting on a Lebesgue probability space $(X, \mathcal{B}, u)$. The action of T on an invariant sub-σ-algebra of $\mathcal{B}$ is called a *factor* of T.

Sinai's weak isomorphism theorem [4] tells us that T has many Bernoulli factors. A question was raised whether the sub-σ-algebras of the Bernoulli factors span the total σ-algebra.

In this note we show that actually one can find three Bernoulli factors that already span $\mathcal{B}$.

LEMMA 1. *Let T be a transformation that splits into a direct product of two factors $\mathcal{B}_1$ and $\mathcal{B}_2$ (i.e., $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$) such that T acting on $\mathcal{B}_1$ is the Bernoulli 2-shift and $\mathcal{B}_2$ is generated by two set partition. Then, there is a Bernoulli factor $\mathcal{B}_3$ such that $\mathcal{B}_3$ is independent of $\mathcal{B}_2$ and such that $\mathcal{B}_1 \vee \mathcal{B}_3 \supset \mathcal{B}_2$.*

PROOF. Let $P = (B_1, B_2)$ be the independent generator for $\mathcal{B}_1$ $\mu(B_1) = \mu(B_2) = \frac{1}{2}$ and $Q = (A_1, A_2)$ a generator for $\mathcal{B}_2$. Consider the partition $R = (C_1, C_2)$ where

$$C_1 = (B_1 \cap A_1) \cup (B_2 \cap A_2).$$

Obviously R is independent partition under $\hat{T}$ and $P \vee R \supset Q$. Also R is independent of $\mathcal{B}_2$.

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LEMMA 2. *Let (A_0, A_0^c) be a partition, independent under T. Let (A_1, A_2) be a partition of A_0^c (i.e., $A_1 \cup A_2 = A_0^c$) which is independent under the induced transformation on A_0^c, $T_{A_0}^c$. Assume also that (A_1, A_2) is independent of the σ-algebra $(A_0, A_0^c)_T \cap A_1$. (If P is a partition $(P)_T = \vee_i T^i P$.) Then, the partition (A_0, A_1, A_2) is independent under T.*

PROOF. Let $i_j \in \{0,1,2\}$, $j = 0, \cdots, n-1$. $j \subset \{0, \cdots, n-1\}$ is the subset of indices such that $i_j = 0$ for $j \in J$. Then

$$\mu\left(\bigcap_{j=0}^{n-1} T^{-j} A_{i_j}\right) = \mu\left(\bigcap_{j\in J} T^{-j} A_0 \cap \bigcap_{j \notin J} T^{-j}(A_{i_j} \cap A_0^c)\right)$$

$$= \mu^{|J|}(A_0)\cdot \mu^{n-|J|}(A_0^c)\mu\left(\bigcap_{j\notin J} T^{-j}A_{i_j} \Big/ \bigcap_{i=J} T^{-j}A_0 \cap \bigcap_{j\neq J} T^{-j}A_0^c\right)$$

$$= \mu^{|J|}(A_0)\cdot \mu^{n-|J|}(A_0^c)\cdot \prod_{j\notin J} \mu(A_{i_j})/\mu(A_0^c)$$

$$= \prod_{j=0}^{n-1} \mu(A_{i_j}).$$

THEOREM 1. *Let $\mathscr{B}_0$ be a factor of full entropy. Let (A_0, A_1, A_2) be a partition such that $\mu(A_0) < 1$. Then, given $\varepsilon > 0$ we can find sets $\tilde{A}_1$ and $\tilde{A}_2$ such that $A_1 \cup A_2 = \tilde{A}_1 \cup \tilde{A}_2$, $\mu(A_1 \Delta \tilde{A}_1) < \varepsilon$ and*

$$\mathscr{B}_0 \vee (A_0, \tilde{A}_1, \tilde{A}_2)_T = \mathscr{B}.$$

(We use the notation, $(P)_T = \vee_{i=-\infty}^{\infty} T^i P$).

The proof of Theorem 1 will follow easily from the following.

APPROXIMATION LEMMA. *Let P be a partition of full entropy. Let $R = (A_0, A_1, A_2)$ be a given partition such that $\mu(A_0) < 1$. Let $Q \supset P$ be another partition. Then, given $\varepsilon > 0$ we can find sets $\bar{A}_1$ and $\bar{A}_2$ such that $A_1 \cup A_2 = \bar{A}_1 \cup \bar{A}_2$, $\mu(A_1 \Delta \bar{A}_1) < \varepsilon$ and $(P)_T \vee (A_0, \bar{A}_1, \bar{A}_2)_T \overset{\varepsilon}{\supset} Q$.*

PROOF. Put $\gamma = \frac{1}{2}(1 - \mu(A_0)) > 0$, $\varepsilon_1 = \varepsilon/4$ and $\delta = \frac{1}{2}\varepsilon_1^2\gamma$. Choose n_0 so large such that $2^{\varepsilon_1^2 \gamma n} > 2^{\delta n}$ for $n > n_0$. Let n_1 be so large that if $m = [\varepsilon_1 \cdot \gamma \cdot n] + 1$, there are $[2^{\varepsilon_1^2 \gamma n}]$ sequences $\xi = (\xi_1, \cdots, \xi_m)\xi_i \in \{1,2\}$ such that the probability of any ξ sequence to appear as m-(A_1, A_2)-name under $T_{A_0^c}$ is less than ε_1, assume also that $\xi_1 = 1$ for all ξ. Using the Shannon–McMillan Theorem, the Ergodic Theorem and Rokhlin Theorem we can find $n > \max(n_0, n_1)$ such that there is a Rokhlin set F, where $F, T^{-1}F, \cdots, T^{-n}F$ are disjoint,

$$\mu\left(\bigcup_{i=0}^{n} T^{-i}F\right) > 1-\varepsilon_1$$

and the following hold.

The partition of F according to n-P-names is such that all (the "good" atoms) but a set of ε_1-fraction of the atoms split into at most $2^{\delta n}$-$Q \vee R$-names. Also, the frequency of visits in $A_1 \cup A_2$ is at list γ.

Now, consider a "good" n-p-atom of F, assign to it a ξ sequence. Change the partition (A_1, A_2) on the first m occurrences of A_0^c, to get $\bar{A}_1, \bar{A}_2$, in such a way that it will have ξ as its m-$(\bar{A}_1, \bar{A}_2)$-name under $T_{A_0^c}$. Also in any later occurrence of the ξ name under $T_{A_0^c}$, change the first entry so that the only occurrences of such sequences in the $(\bar{A}_1, \bar{A}_2)$-name will be at the beginning. Now, it is easy to see that

$$\bigvee_{i=-2n}^{0} T^i(P \vee \bar{R}) \overset{4\varepsilon_1}{\supset} Q$$

and that $\mu(A_1 \Delta \bar{A}_1) < 4\varepsilon_1$.

Proof of Theorem 1. Let Q_n, $n = 1, 2, \cdots$ be a sequence of finite partitions such that $Q_n \subset Q_{n+1}$ and $\bigvee_{n=1}^{\infty} Q_n = \mathscr{B}$.

Define inductively the sets A_1^n and A_2^n as follows:

(1) $A_1^0 = A_1$, $A_2^0 = A_2$.

(2) Let A_1^n be such that $\mu(A_1^n \Delta A_1^{n-1}) < \varepsilon/2^n$ and $\mathscr{B}_0 \vee (A_0, A_1^n, A_2^n) \overset{2^{1/(n+1)}}{\supset} Q_n$, and there are natural numbers $N_1, \cdots, N_{n-1}$ such that

$$\mathscr{B}_0 \vee \bigvee_{i=-N_j}^{N_j} T^i(A_0, A_1^n, A_2^n) \overset{1/2^j+\cdots+1/2^{j+n}}{\supset} Q_j$$

for $j = 1, \cdots, n-1$.

It follows that there is a natural number N_n, such that

$$\mathscr{B}_0 \vee \bigvee_{i=-N_n}^{N_n} T^i(A_0, A_1^n, A_2^n) \overset{1/2^n}{\supset} Q_n.$$

Use the approximation lemma to find a set A_1^{n+1} such that $\mu(A_1^n \Delta A_1^{n+1}) < \varepsilon/2^{n+1}$ and A_1^{n+1} is so close to A_1^n that for $j = 1, \cdots, n$

$$\mathscr{B}_0 \vee \bigvee_{i=-N_j}^{N_j} T^i(A_0, A^{n+1}, A_2^{n+1}) \overset{1/2^j+1/2^{j+1}+\cdots+1/2^{j+n}}{\supset} Q_j.$$

The sets A_1^n converge to $\tilde{A}_1$ and

$$\mathscr{B}_0 \vee \bigvee_{i=-N_j}^{N_j} T^i(A_0, \tilde{A}_1, \tilde{A}_2) \overset{1/2^{j-1}}{\supset} Q_j.$$

THEOREM 2. *Let T be an invertible ergodic measure preserving transformation of positive entropy acting on $(X, \mathcal{B}, \mu)$ and $\varepsilon > 0$. Then, there are three Bernoulli factors $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_0 \vee \mathcal{B}_1 \vee \mathcal{B}_2 = \mathcal{B}$ and $h(T, \mathcal{B}_1) = h(T, \mathcal{B}_2) < \varepsilon$.*

PROOF. Choose $\mathcal{B}_0$ to be a Bernoulli factor of full entropy. Let $P = (A_0, A_o^c)$ be an independent partition under T such that $h(T) - h(P) - \mu(A_0^c) = \gamma > 0$. It follows from Theorem 1 that there is a two set partition of A_0^c, (A_1, A_2) such that

$$h\left(\frac{\mu(A_1)}{\mu(A_0^c)}, \frac{\mu(A_2)}{\mu(A_0^c)}\right) < \gamma, \qquad (\bar{P})_T \vee \mathcal{B}_0 = \mathcal{B},$$

where $\bar{P} = (A_0, A_1 A_2)$; since

$$h(T) - h(\bar{P}) - \mu(A_0^c) > 0$$

we have

$$\frac{h(T)}{\mu(A_0^c)} - \frac{h(\bar{P})}{h(A_0^c)} > 1 = \log 2$$

(we take the log to the base 2).

It follows from Abramov's formula [1] and the relativized version of Sinai's theorem of [2] that there is a two set partition (B_1, B_2) of A_0^c such that $\mu(B_1) = \mu(B_2)$ and (B_1, B_2) is independent under $T_{A_0^c}$ and is independent of $(P)_T \cap A_0^c$. Use Lemma 1 to find a partition (C_1, C_2) which is independent under $T_{A_0^c}$ and independent of (A_1, A_2) and together with (B_1, B_2) spans (A_1, A_2). Now put $\mathcal{B}_1 = (A_0, B_1, B_2)$ and $\mathcal{B}_2 = (A_0, C_1, C_2)$.

REMARK 1. We say that T has the Weak Pinsker Property (WPP) (see [3]) if for given $\varepsilon > 0$, $T = T_1 \times T_\varepsilon$ where T_1 is Bernoulli and $h(T_\varepsilon) < \varepsilon$. It can be shown that in the case where T has the WWP one can find two Bernoulli factors that span T. It is not known whether the WPP is universal. All the known examples were checked to have the WPP. Therefore, the problem whether for each transformation we can find two Bernoulli factors that span it is an intermediate problem to the WPP problem.

REMARK 2. It is known that every Bernoulli transformation T is determined by three invariant sub-σ-algebras. Namely, any other transformation that leaves those σ-algebras invariant must be a power of T. Combining these results with Theorem 1 we have:

THEOREM 3. *For every ergodic transformation T of positive entropy there are nine invariance sub-σ-algebras such that every transformation that leaves those σ-algebras invariant must be T^n for some n in Z.*

Again, it is an open problem to determine the minimal number in Theorem 3.

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